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Zhang Suhua[a]; An Hailong[a]; Guan Ronghua[b]; Yang Guochen[a]
[a] School of Science, Hebei University of Technology, Tianjin 300130, PR China [b] Applied Physics Institute of North China Electric Power University, Baoding 071003, PR China

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The bistable state of a twisted nematic liquid crystal cell with weak anchoring boundary

ZHANG SUHUA*†, AN HAILONG†, GUAN RONGHUA‡ and YANG GUOCHEN†

†School of Science, Hebei University of Technology, Tianjin 300130, PR China
‡Applied Physics Institute of North China Electric Power University, Baoding 071003, PR China

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On the basis of the modified general Rapini–Papoular expression for the anchoring energy, a twisted nematic liquid crystal cell has been studied analytically. In this paper, a new variable is introduced and is suitable for the calculation of the threshold point and the saturation point. The free energy being smallest in the equilibrium state, we find that bistable states can be formed from the uniform twisted state plus the disturbed state, the disturbed state plus the saturation state, and the uniform twisted state plus the saturation state.

1. Introduction

Surface effects of liquid crystals are important for both device application and basic understanding of physical phenomena. Surface-induced alignment has long been used to obtain the uniform alignment of a nematic liquid crystal (NLC) for practical as well as measurement purposes. However the mechanism of the director alignment of a liquid crystal (LC) film by the substrate surface is not well understood. To quantify how strongly a NLC is oriented or anchored, the interfacial free energy g_s, also called anchoring energy, has been introduced. Rapini and Papoular have proposed a simple phenomenological expression for the anchoring energy per unit area [1]:

$$g_s = \frac{1}{2} A \sin^2 \theta \tag{1}$$

where θ is the angle between the easy direction **e** and the director **n** of the NLC at the nematic–wall interface, and A is the anchoring strength. This is the so-called RP formula.

The RP formula describes many effects successfully in the presence of a surface. However, it is found that results calculated from the RP formula do not agree well with the experimental observations in some cases (for example, the distortions of the director in strong external fields)[2]. Many authors have introduced new anchoring energy forms to replace the RP formula: Yang *et al.* express g_s in Legendre polynomial functions of $\sin \theta$ [3–6]. Barbero *et al.* expand g_s into a Fourier series [7]. Barnik *et al.* [8] utilize the elliptic function of θ as the functional form of g_s [8]. The lowest order series modification of equation (1) can be expressed as

$$g_s = A \sin^2 \theta \left(1 + \zeta \sin^2 \theta\right) \tag{2}$$

where ζ is a modified parameter. This form has been investigated experimentally by some authors [4, 8], and at present is generally accepted [9].

We now follow Jérŏme [10] and generalize this formula to the two-dimensional case:

$$g_s = -\frac{A}{2} (n \cdot e)^2 \tag{3}$$

which is a nonlinear combination of the azimuthal and polar angles and is called the general RP expression for the anchoring energy. With loss of generality, we introduce a parameter ζ' in equation (3) as the modified RP formula:

$$g_s = -\frac{A}{2} (n \cdot e)^2 \left[1 + \zeta' (n \cdot e)^2\right] \tag{4}$$

which is called the modified general RP expression. We shall mainly use this formula of interfacial potential in this paper.

A great deal of work has been done on the study of the physical effects of weak anchoring liquid crystal cells. It was recently proved that a first order Fréedericksz transition would appear in weak anchoring NLC cells [11, 12]. By means of a LC light valve, an experiment has been realized to render a Fréedericksz transition over a large area nematic film [13].The first order transition is always related to the bistable state,

*Corresponding author. Email: zhangsuhua76@163.com

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which is helpful in the design of bistable twisted nematic liquid crystal displays. According to statistical thermodynamics, the free energy G of the stable state is the smallest among all solutions; others relate to the metastable state. If there are two solutions that together make G the smallest (they have the same value), then the bistable state is formed. Others have derived the threshold strength and saturation strength of the TNLC cell with weak anchoring boundaries [14]. But an unusual case emerges in the results, that the threshold strength may be lagrer than or equal to the saturation strength, and this may lead to the bistable state. In this paper, we will study the bistable state from the review the free energy.

On the basis of the modified general Rapini–Papoular expression for the anchoring energy, we study the property of the bistable state for the weak anchoring TNLC cell analytically. In § 2, we obtain fundamental equations and discuss their solutions. In § 3, a new variable is introduced, and the fundamental equations are rewritten. In § 4, supposing the transition is of second order, the threshold field strength and the saturation field strength are calculated; in § 5, comparing the free energy of every solution, we find that the bistable state can be formed from the uniform twisted state plus the disturbed state, the disturbed state plus the saturation state and the uniform twisted state plus the saturation state. When the bistable state is formed, the threshold field strength or the saturation strength is calculated and is compared with literature the results [14].

2. Fundamental equations and solutions

For a TNLC cell, two substrates lie in the $Z=0$ and $Z=l$ planes. Assuming the two substrates are identical, the easy direction **e** in the $Z=0$ plane is parallel to the direction of the X-axis, and the easy direction **e** in the $Z=l$ plane is parallel to the direction of the Y-axis. The surface energy takes the form of equation (4). The applied field is parallel to the Z axis.

The easy direction **e** and the director **n** can be written as follows:

$$e_{\text{down}}=(1,\ 0,\ 0) \tag{5}$$

$$e_{\text{up}}=(0,\ 1,\ 0) \tag{6}$$

$$n=(\cos\theta\cos\phi,\ \cos\theta\sin\phi,\ \sin\theta) \tag{7}$$

where θ is the tilt angle and ϕ is the azimuthal angle, which are functions of Z, written as $\theta(z)$ and $\phi(z)$. The Gibbs free energy per unit volume in the cell is given by

$$G_{\text{b}}=\frac{1}{2}f(\theta)\left(\frac{\mathrm{d}\theta}{\mathrm{d}z}\right)+\frac{1}{2}h(\theta)\left(\frac{\mathrm{d}\phi}{\mathrm{d}z}\right)^2-\frac{1}{2}\chi_{\text{a}}H^2\sin^2\theta \tag{8}$$

where

$$f(\theta)=K_{11}\left(1+\gamma_1\sin^2\theta\right) \tag{9}$$

$$h(\theta)=\cos^2\theta K_{22}\left(1+\gamma_2\sin^2\theta\right) \tag{10}$$

and $\gamma_1=(K_{33}-K_{11})/K_{11}$, $\gamma_2=(K_{33}-K_{22})/K_{22}$, K_{11}; K_{22} and K_{33} are Frank splay, twist and bend elastic constants, and χ_{a} is the magnetic anisotropy of the TNLC medium. The surface energy per unit area can be expressed as:

$$g_{\text{s}}|_{z=0}=-\frac{A}{2}\cos^2\theta_0\cos^2\phi_0\left(1+\zeta'\cos^2\theta_0\cos^2\phi_0\right) \tag{11}$$

$$g_{\text{s}}|_{z=i}=-\frac{A}{2}\cos^2\theta_l\sin^2\phi_l\left(1+\zeta'\cos^2\theta_l\sin^2\phi_l\right) \tag{12}$$

where θ_0 and ϕ_0, θ_l and ϕ_l are the value of θ and ϕ at $Z=0$ and $Z=l$, respectively. The total energy of the system is

$$\begin{aligned}G=S&\int_0^l\left[\frac{1}{2}f(\theta)\left(\frac{\mathrm{d}\theta}{\mathrm{d}z}\right)^2+\frac{1}{2}h(\theta)\left(\frac{\mathrm{d}\phi}{\mathrm{d}z}\right)^2-\frac{1}{2}\chi_{\text{a}}H^2\sin^2\theta\right]\mathrm{d}z\\&-S\frac{A}{2}\cos^2\theta_0\cos^2\phi_0\left(1+\zeta'\cos^2\theta_0\cos^2\phi_0\right)\\&-S\frac{A}{2}\cos^2\theta_l\sin^2\phi_l\left(1+\zeta'\cos^2\theta_l\sin^2\phi_l\right)\end{aligned} \tag{13}$$

where S is the area of the substrate.

Applying the calculus of variations [15], we obtain equations for θ and ϕ:

$$\begin{aligned}f(\theta)\frac{\mathrm{d}^2\theta}{\mathrm{d}z^2}+\frac{1}{2}\frac{\mathrm{d}f(\theta)}{\mathrm{d}\theta}\left(\frac{\mathrm{d}\theta}{\mathrm{d}z}\right)^2-\frac{1}{2}\frac{\mathrm{d}h(\theta)}{\mathrm{d}\theta}\left(\frac{\mathrm{d}\phi}{\mathrm{d}z}\right)^2\\+\chi_{\text{a}}H^2\sin\theta\cos\theta=0\end{aligned} \tag{14}$$

$$h(\theta)\frac{\mathrm{d}\phi}{\mathrm{d}z}=C_1 \tag{15}$$

where

$$C_1=A\cos^2\theta_0\cos\phi_0\sin\phi_0\left(1+2\zeta'\cos^2\theta_0\cos^2\phi_0\right). \tag{16}$$

The boundary conditions are

$$f(\theta_0)\left(\frac{\mathrm{d}\theta}{\mathrm{d}z}\right)\Bigg|_{z=0}=\begin{aligned}&A\cos\theta_0\sin\theta_0\cos^2\phi_0\\&\left(1+2\zeta'\cos^2\theta_0\cos^2\phi_0\right)\end{aligned} \tag{17}$$

$$h(\theta_0)\left(\frac{\mathrm{d}\phi}{\mathrm{d}z}\right)\Big|_{z=0}=A\cos^2\theta_0\cos\phi_0\sin\phi_0 \left(1+2\zeta'\cos^2\theta_0\cos^2\phi_0\right) \tag{18}$$

The equations (14) and (15) with the boundary conditions (17) and (18) have three solutions, of which two are trivial. The solutions and corresponding Gibbs free energy are as follows:

(1) Uniform twisted solution: $\theta\equiv 0$

$$G_\mathrm{u}=S\frac{\pi K_{22}}{\lambda l}\left[\frac{\pi}{\lambda}\cos^2\phi_0\sin^2\phi_0\left(1+2\zeta'\cos^2\phi_0\right)^2 -2\cos^2\phi_0\left(1+\zeta'\cos^2\phi_0\right)\right] \tag{19}$$

$$\phi(z)=\phi_0+\frac{A\cos\phi_0\sin\phi_0\left(1+2\zeta'\cos^2\phi_0\right)}{K_{22}}z \tag{20}$$

where ϕ_0 satisfies

$$\frac{\frac{\pi}{2}-2\phi_0}{\cos\left(\frac{\pi}{2}-2\phi_0\right)\left(1+2\zeta'\cos^2\phi_0\right)}=\frac{\pi}{2\lambda} \tag{21}$$

and λ is the reduced anchoring energy strength, written as $\lambda=\pi K_{22}/Al$.

(2) Saturation solution: $\theta\equiv\pi/2$

$$G_\mathrm{s}=-\frac{1}{2}\chi_\mathrm{a}H^2lS. \tag{22}$$

(3) Disturbed solution: $\theta=\theta(z)$, $\phi=\phi(z)$, where θ and ϕ satisfy, respectively:

$$\frac{1}{2}=\int_{\theta_0}^{\theta_\mathrm{m}}\sqrt{N(\theta)}\mathrm{d}\theta \tag{23}$$

$$\frac{\pi}{4}-\phi_0=\int_{\theta_0}^{\theta_\mathrm{m}}\frac{C_1}{h(\theta)}\sqrt{N(\theta)}\mathrm{d}\theta \tag{24}$$

with the boundary condition:

$$f(\theta_0)\frac{1}{\sqrt{N(\theta_0)}}=A\cos\theta_0\sin\theta_0\cos^2\phi_0 \left(1+2\zeta'\cos^2\theta_0\cos^2\phi_0\right) \tag{25}$$

where

$$N(\theta)=\frac{f(\theta)}{\chi_\mathrm{a}H^2\left(\sin^2\theta_\mathrm{m}-\sin^2\theta\right)+C_1^2\left[\frac{1}{h(\theta_\mathrm{m})}-\frac{1}{h(\theta)}\right]} \tag{26}$$

$$N(\theta_0)=N(\theta)|_{\theta=\theta_0} \tag{27}$$

$$G_\mathrm{d}=S\frac{1}{2}\chi_\mathrm{a}H^2\left(\sin^2\theta_\mathrm{m}l-4\int_0^{\frac{1}{2}}\sin^2\theta\,\mathrm{d}z\right)+\frac{C_1^2lS}{2h(\theta_\mathrm{m})} -AS\cos^2\theta_0\cos^2\phi_0\left(1+\zeta'\cos^2\theta_0\cos^2\phi_0\right) \tag{28}$$

and θ_m is the value of θ at $Z=1/2$.

We now discuss the stable solution, with the application of an external magnetic field $\boldsymbol{H}$ increasing from 0. If $\boldsymbol{H}$ is small, G_u is the smallest among the three Gibbs free energy values, and the stable solution is the uniform twisted solution (19). As $\boldsymbol{H}$ increases continuously and beomes equal to or bigger than a critical value, G_θ is smaller, and the stable solution is the disturbed solution (28); $\boldsymbol{H}$ is then at the threshold field strength, $\boldsymbol{H}_\mathrm{th}$. When $\boldsymbol{H}$ is equal to or bigger than the critical value of $\boldsymbol{H}$, G_s is the smallest, and the stable solution is the saturation solution (22). $\boldsymbol{H}$ is then at saturation field strength, $\boldsymbol{H}_\mathrm{sat}$. From this, we can determine the transition property of each state.

3. Variable transform

We introduce a new state variable, convenient for the calculation of the threshold state and saturation state. Defining parameters and variables;

$$u=\sin^2\theta_\mathrm{m},\ v=\frac{\tan^2\theta}{\tan^2\theta_\mathrm{m}},\ v_0=\frac{\tan^2\theta_0}{\tan^2\theta_\mathrm{m}}. \tag{29}$$

The reduced field strength h, and reduced free energy g are:

$$h=\frac{H}{H_\mathrm{c}^0},\quad H_\mathrm{c}^0=\frac{\pi}{l}\left(\frac{K_{11}}{\chi_\mathrm{a}}\right)^{\frac{1}{2}} \tag{30}$$

$$g=\frac{lG}{K_{11}S}. \tag{31}$$

By means of these parameters and variables, the equations (23)–(26) can be rewritten as

$$\frac{\pi}{2}h=\int_{v_0}^1\left(\frac{1-u+uv+\gamma_1uv}{1-v}\right)^{\frac{1}{2}}\left(\frac{1}{1+X}\right)\frac{1}{2}\frac{1}{v^{\frac{1}{2}}(1-u+uv)}\mathrm{d}v \tag{32}$$

$$\frac{\left(\frac{\pi}{2}-2\phi_0\right)(1-u+uv_0)^2}{\cos\left(\frac{\pi}{2}-2\phi_0\right)\left[1-u+uv_0+2\zeta'(1-u)\cos^2\phi_0\right]}=\frac{1}{h\lambda} \times\int_{v_0}^1\frac{(1-u+uv)^2}{(1-u+uv+\gamma_2uv)}\left(\frac{1-u+uv+\gamma_1uv}{1-v}\right)^{\frac{1}{2}} \left(\frac{1}{1+X}\right)^{\frac{1}{2}}\frac{1}{2}\frac{1}{v^{\frac{1}{2}}(1-u+uv)}\mathrm{d}v \tag{33}$$

$$h=\frac{K_{22}}{K_{11}}\frac{1}{\lambda}\frac{\cos^2\phi_0\left[1-u+uv_0+2\zeta'(1-u)\cos^2\phi_0\right]v_0^{\frac{1}{2}}}{(1-u+uv_0)(1-u+uv_0+\gamma_1 uv_0)^{\frac{1}{2}}(1-v_0)(1+X_0)^{\frac{1}{2}}} \quad (34)$$

where

$$X=\frac{K_{22}}{K_{11}}\frac{1}{\lambda^2h^2}\frac{\cos^2\phi_0\sin^2\phi_0\left[1-u+uv_0+2\zeta'(1-u)\cos^2\phi_0\right]^2}{(1-u+uv_0)^4}\times\frac{(1-u+uv)\left[1-u+uv+\gamma_2uv-(1+\gamma_2u)(1-u+uv)^2\right]}{u(1-v)(1+\gamma_2u)(1-u+uv+\gamma_2uv)} \quad (35)$$

$$X_0=X|_{v=v_0}. \quad (36)$$

The reduced free energy of each solution can be rewritten as:

$$g_{\mathrm{u}}=\frac{1}{2}\frac{K_{22}}{K_{11}}\frac{\pi}{\lambda}\left[\begin{array}{l}\frac{\pi}{\lambda}\cos^2\phi_0\sin^2\phi_0\left(1+2\zeta'\cos^2\phi_0\right)^2\\ -2\cos^2\phi_0\left(1+\zeta\cos^2\phi_0\right)\end{array}\right] \quad (37)$$

$$g_{\mathrm{s}}=-\frac{1}{2}\pi^2h^2 \quad (38)$$

$$g_d=\frac{h\pi^2u}{2}\left(h-\frac{4}{\pi}I_2\right)+\frac{\pi^2}{\lambda^2}\frac{K_{22}}{K_{11}}\frac{(1-u)\cos^2\left(\frac{\pi}{2}-\phi_0\right)\left[1-u+uv_0+2\zeta'(1-u)\cos^2\phi_0\right]^2}{8(1-u+uv_0)^4(1+\gamma_2u)}-\frac{\pi}{\lambda}\frac{K_{22}}{K_{22}}\frac{1-u}{(1-u+uv_0)^2}\cos^2\phi_0\left[1-u+uv_0+\zeta'(1-u)\cos^2\phi_0\right]. \quad (39)$$

We now define several integrals:

$$I_1=\int_{v_0}^1 M(v)\mathrm{d}v \quad (40)$$

$$I_2=\int_{v_0}^1\frac{v}{1-u+uv}M(v)\mathrm{d}v \quad (41)$$

$$I_3=\int_{v_0}^1\frac{(1-u+uv)^2}{1-u+uv+\gamma_2uv}M(v)\mathrm{d}v \quad (42)$$

where

$$M(v)=\left(\frac{1-u+uv+\gamma_1uv}{1-v}\right)^{\frac{1}{2}}\left(\frac{1}{1+X}\right)^{\frac{1}{2}}\frac{1}{2\quad v^{\frac{1}{2}}(1-u+uv)} \quad (43)$$

equations (30) and (31) can then be rewritten as:

$$\frac{\pi}{2}h=I_1 \quad (44)$$

$$\frac{\left(\frac{\pi}{2}-2\phi_0\right)(1-u+uv_0)^2}{\cos\left(\frac{\pi}{2}-2\phi_0\right)\left[1-u+uv_0+2\zeta'(1-u)\cos^2\phi_0\right]}=\frac{1}{h\lambda}I_3. \quad (45)$$

Based on the equations (32)–(34), for a given u, we can solve v_0, ϕ_0 and h; we can then calculate the free energy of each solution and compare their values. The most stable state can be decided and the bistable state also determined.

4. The threshold state and the saturation state of the second order transition

4.1. *The threshold field strength for the second order transition*

We suppose that the transition from the uniform twisted state to the disturbed state is of second order and the transition is continuous, and we discuss the threshold field strength h_{th} and the saturation field h_{sat}. Fistly we consider h_{th}; the threshold field strength can be easily obtained from the equations and boundary conditions. When u=0, from the equations (32)–(34), we obtain:

$$\frac{\pi}{2}h_{\mathrm{th}}=\left(\frac{1}{1+X_{\mathrm{th}}}\right)^{\frac{1}{2}}\int_{v_0}^1\frac{1}{2[v(1-v)]^{\frac{1}{2}}}\mathrm{d}v \quad (46)$$

$$\frac{\left(\frac{\pi}{2}-2\phi_0\right)}{\cos\left(\frac{\pi}{2}-2\phi_0\right)(1+2\zeta'\cos^2\phi_0)}=\frac{1}{h_{\mathrm{th}}\lambda}\left(\frac{1}{1+X_{\mathrm{th}}}\right)^{\frac{1}{2}}\int_{v0}^1\frac{1}{2[v(1-v)]}\mathrm{d}v \quad (47)$$

$$h_{\mathrm{th}}=\frac{K_{22}}{K_{11}}\frac{1}{\lambda}\frac{\cos^2\phi_0\left(1+2\zeta'\cos^2\phi_0\right)v_0^{\frac{1}{2}}}{(1-v_0)^{\frac{1}{2}}(1+X_{\mathrm{th}})^{\frac{1}{2}}}. \quad (48)$$

Equation (35) can be rewritten as

$$X_{\mathrm{th}}=\frac{K_{22}}{K_{11}}\frac{\cos^2\phi_0\sin^2\phi_0\left(1+2\zeta'\cos^2\phi_0\right)^2}{\lambda^2h_{\mathrm{th}}^2}(1-\gamma_2). \quad (49)$$

From equations (46) and (47), we obtain:

$$\frac{\left(\frac{\pi}{2}-2\phi_0\right)}{\cos\left(\frac{\pi}{2}-2\phi_0\right)(1+2\zeta'\cos^2\phi_0)}=\frac{\pi}{2\lambda} \quad (50)$$

and equation (48) yields

$$\left(\frac{v_0}{1-v_0}\right)^{\frac{1}{2}}=\tan\left[\frac{\pi}{2}h_{\mathrm{th}}(1+X_{\mathrm{th}})^{\frac{1}{2}}\right] \quad (51)$$

substituting equation (51) into the equation (48), gives:

$$h_{\rm th}=\frac{K_{22}}{K_{11}}\frac{1}{\lambda}\frac{\cos^2\phi_0\left(1+2\zeta'\cos^2\phi_0\right)}{(1+X_{\rm th})^{\frac{1}{2}}}\tan\left[\frac{\pi}{2}h_{\rm th}(1+X_{\rm th})^{\frac{1}{2}}\right]. \tag{52}$$

We can then solve the threshold field from equations (49), (50) and (52).

4.2. *The saturation field strength for the second order transition*

We now consider the saturation field strength $h_{\rm sat}$. From equations (32)–(34), let $\theta_{\rm m}\rightarrow\frac{\pi}{2}$, giving:

$$\frac{\pi}{2}h_{\rm sat}=(1+\gamma_1)^{\frac{1}{2}}\int_{v_0}^{1}\left(\frac{1}{1+X_{\rm sat}}\right)^{\frac{1}{2}}\frac{1}{2v[(1-v)]^{\frac{1}{2}}}\,{\rm d}v \tag{53}$$

$$\frac{\left(\frac{\pi}{2}-2\phi_0\right)v_0}{\cos\left(\frac{\pi}{2}-2\phi_0\right)}=\frac{(1+\gamma_1)^{\frac{1}{2}}}{h_{\rm sat}\,\lambda(1+\gamma_2)}\int_{v_0}^{1}\left(\frac{1}{1+X_{\rm sat}}\right)^{\frac{1}{2}}\frac{1}{2(1-v)^{\frac{1}{2}}}\,{\rm d}v \tag{54}$$

$$h_{\rm sat}=\frac{K_{22}}{K_{11}}\frac{1}{\lambda}\frac{\cos^2\phi_0}{(1+\gamma_1)^{\frac{1}{2}}(1+X_{0\rm sat})^{\frac{1}{2}}(1-v_0)^{\frac{1}{2}}} \tag{55}$$

where

$$X_{\rm sat}=\frac{K_{22}}{K_{11}}\frac{\cos^2\phi_0\sin^2\phi_0 v}{\lambda^2h_{\rm sat}^2v_0^2(1+\gamma_2)} \tag{56}$$

$$X_{0\,\rm sat}=X_0|_{u=1} \tag{57}$$

From these equations, we can calculate the saturation field.

4.3. *The calculation results*

Figures 1–3 show the results of numerical calculations in which we take $K_{33}/K_{11}=1.5$; $K_{22}/K_{11}=0.6$ [14]. From these three figures, it can be seen that the threshold field may be larger than the saturation field, although this case is unusual. In figure 1, where $\zeta'=0.2$, the unusual case emerges when $\lambda>0.77$; in figure 2, where $\zeta'=0$, the unusual case emerges when $\lambda>1.07$. This is the same as results in the literature [13]. When $\zeta'=-0.2$, the saturation field strength is larger than the threshold field strength, the usual situation. The proposition that all transitions are of second order leads to, meaningless calculation results.

5. The bistable state

In principal, when $\boldsymbol{H}$ increases from zero, the nature of the transition from the uniform twisted solution to the disturbed solution to the saturation solution can be judged by comparing the free energy of each solution. If the free energy of two solutions is equal and is the smallest, but u is different, the two states form a bistable state. We will study the nature of the transition for different values of the reduced anchoring energy λ.

5.1. *The reduced anchoring energy is very small, λ≈0.5*

From figure 4 (*a*) we can see that one magnetic field value h corresponds to two values of u; and this indicates the disturbed solution, where $g_{\rm d}$ has two solutions for a given magnetic field h. From figure 4 (*b*) we see that when the magnetic field strength h is equal to the threshold field strength $h_{\rm th}$, the free energy of the

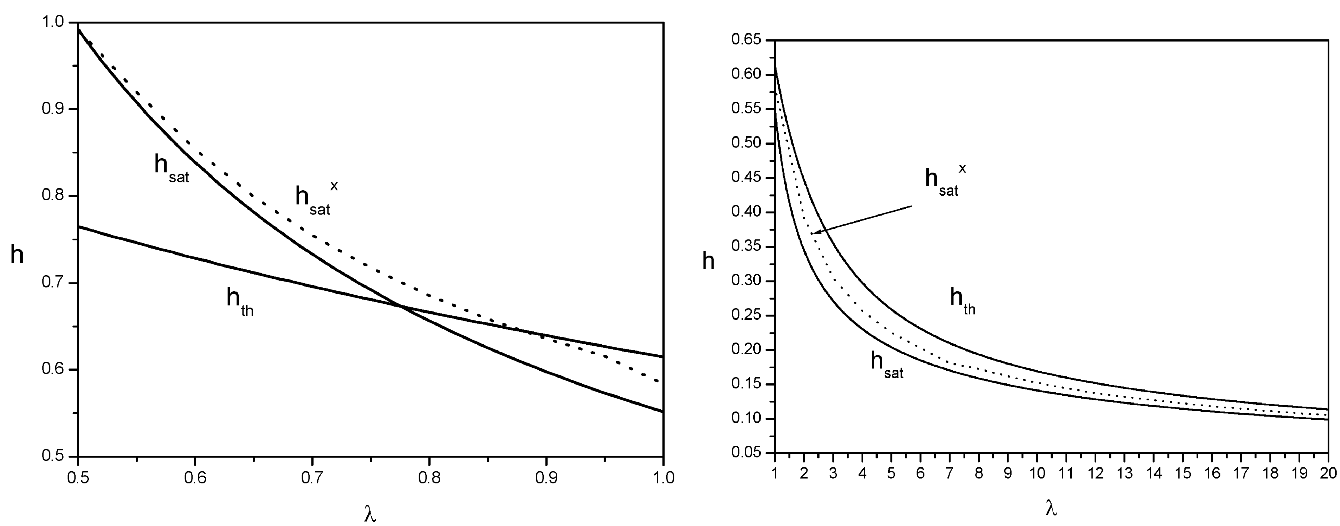


Figure 1. Threshold field $h_{\rm th}$ and saturation field $h_{\rm sat}$ versus the reduced anchoring strength λ; $\zeta'=0.2$.

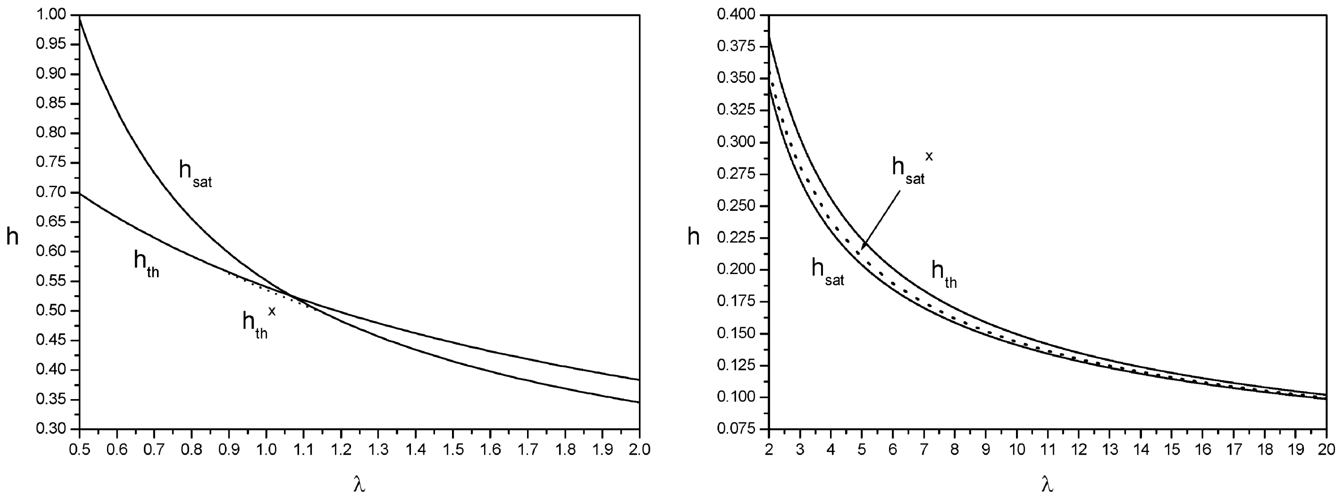


Figure 2. Threshold field h_{th} and saturation field h_{sat} versus the reduced anchoring strength λ; $\zeta'=0$.

uniform twisted solution g_u is equal to that of the disturbed solution g_d, and the director is parallel to the substrates, $u=0$. As h increases, the free energies of the disturbed solution g_d and the saturation solution g_s decrease at same time; when $h_{th}<h<0.799257$, g_d is at a minimum and the disturbed state is the stable state, when $h=0.799257$, g_d is equal to g_s. For the disturbed solution, $u=0.78$, and for the saturation solution, $u=1$, the transition from the disturbed state to the saturation state occurs at this point and the two states form the bistable state. The transition is sudden and is a first order transition; the corresponding magnetic field strength h is termed the saturation field strength of the first order transition, written as h^{x}_{sat}.

From the figure 5 we can see that the threshold field strength h_{th} is smaller than the saturation field strength h_{sat}; as h increase, the deviation of the director increases continuously. When $u=0$, the field strength is the threshold strength h_{th}, and the free energy of the uniform twisted solution g_u is equal to that of the disturbed solution g_d. A second order transition occurs between the uniform twisted and the disturbed solutions; when $u=1$, the field strength is the saturation strength h_{sat}, and g_d is equal to g_s. The second order transition h occurs at this point.

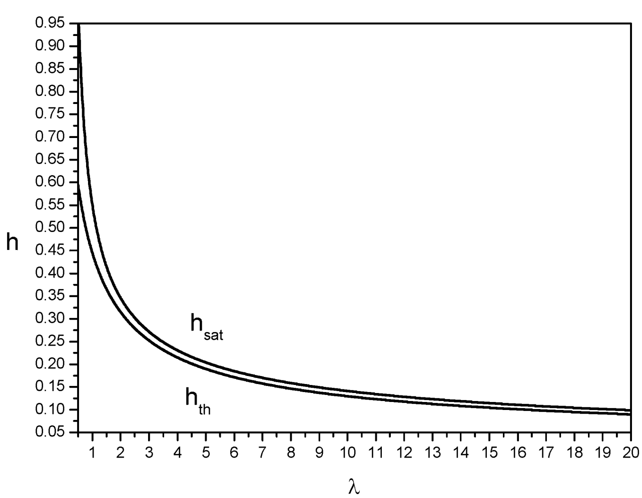


Figure 3. Threshold field h_{th} and saturation field h_{sat} versus the reduced anchoring strength λ; $\zeta=-0.2$.

For $\zeta'=-0.2$ and $\lambda=0.6$, a similar calculation produces results the same as for figure 5.

5.2. *The reduced anchoring energy* $\lambda\approx 1$

The result depicted in figure 6 is analogous with that shown in figure 4. When $h=0.704056$, the free energy of the disturbed solution g_d is equal to that of the saturation solution g_s. For the disturbed solution, $u=0.56$, and for the saturation solution, $u=1$; they form the bistable state.

From figure 7 can be seen that, as u increases, h first decreases and then increases. When $h<0.517328$, the free energy of the uniform twisted solution g_u is the smallest and the uniform twisted state is the stable state; the director is parallel to the substrate and $u=0$, when $h=0.517328$, g_u is equal to g_d; for the disturbed solution, $u=0.64$. The uniform twisted state and the disturbed state form a bistable state. The corresponding field strength is termed the threshold strength of the first order, written as h^{x}_{th}.

For $\zeta'=-0.2$ and $\lambda=1$, the results are similar to those of figure 5. The director deviates from the substrate continuously, and the transition between the uniform twisted state, the disturbed state and the saturation state is continuous.

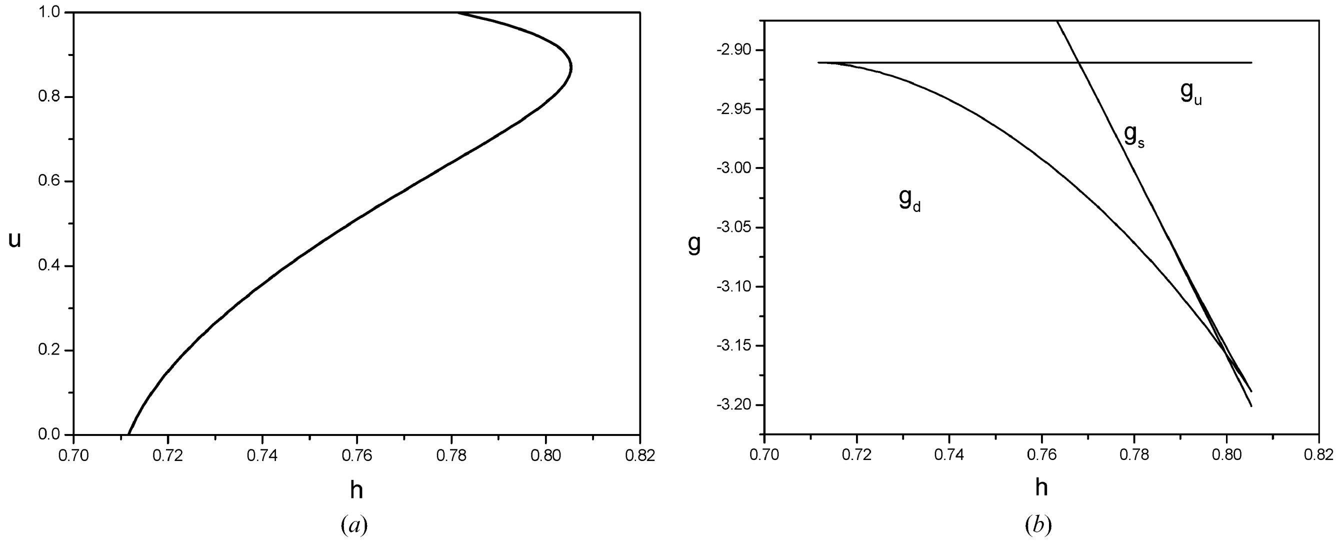


Figure 4. Plots of (*a*) u versus h and (*b*) g versus h; $\zeta'=0.2$, $\lambda=0.65$.

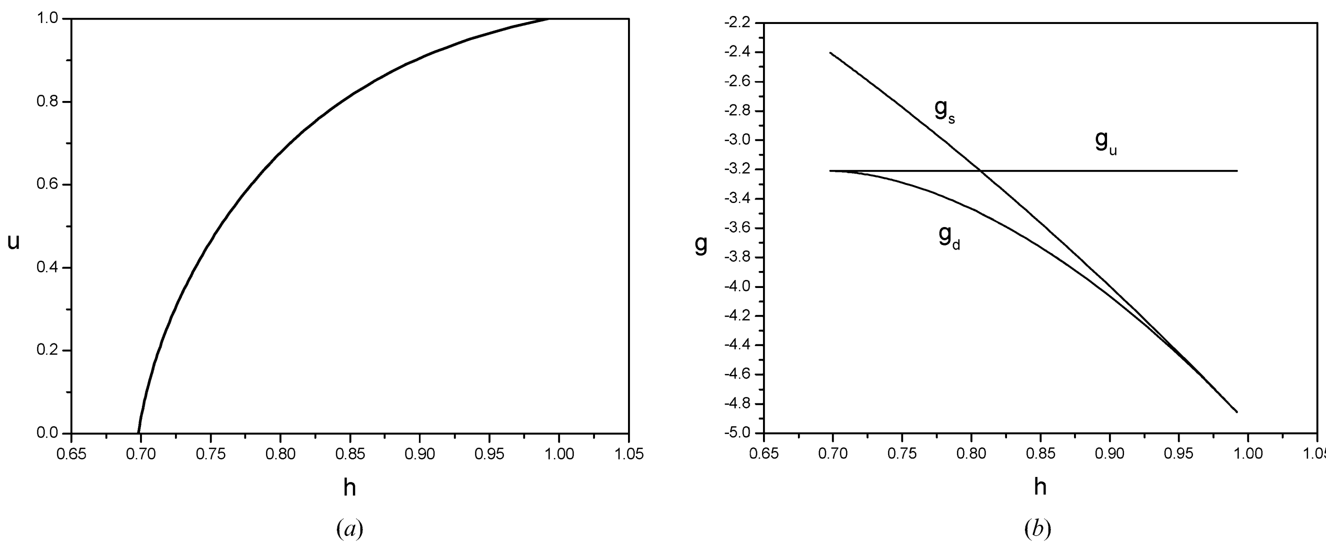


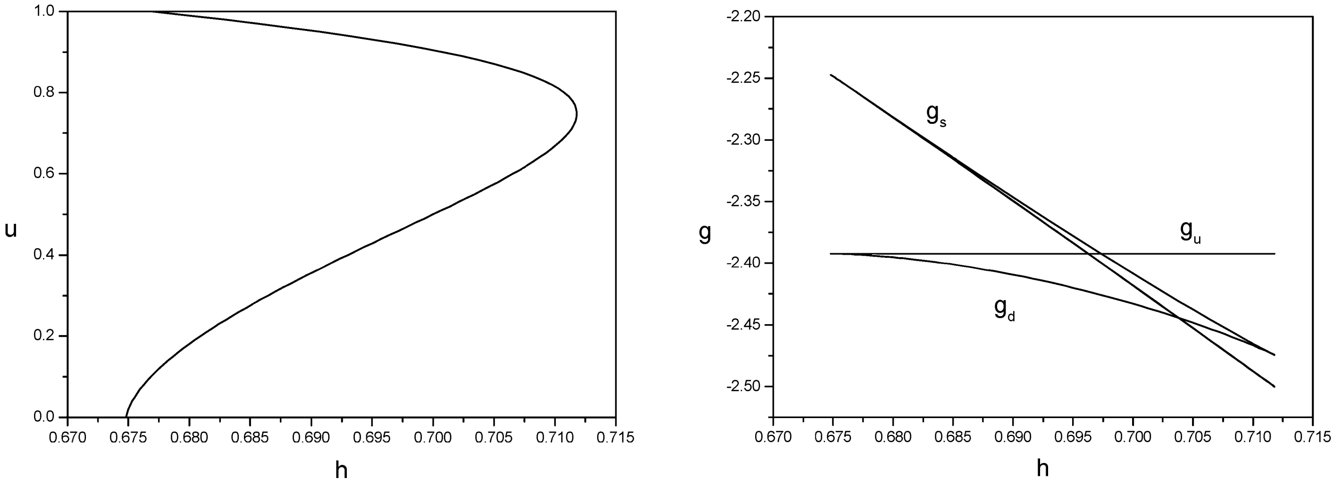
Figure 5. Plots of (*a*) u versus h and (*b*) g versus h; $\zeta'=0$, $\lambda=0.5$.

5.3. *The reduced anchoring energy is very large, $\lambda \approx 20$*

From figure 8 it can be seen that as u increases, h decreases. When $h<0.105154$, the uniform twisted state is the most stable; as h increases, the free energies of the disturbed solution g_d and the saturation solution g_s both decrease, but the saturation solution decrease more rapidly. When $h=0.105154$, the free energy of the uniform twisted solution g_u is equal to g_s; for the uniform twisted state, $u=0$, but for the saturation state, $u=1$.The uniform twisted state and the saturation state form the bistable state. The change of the director is sudden. The field strength is the saturation field of the first order. In this situation, the threshold field strength is meaningless

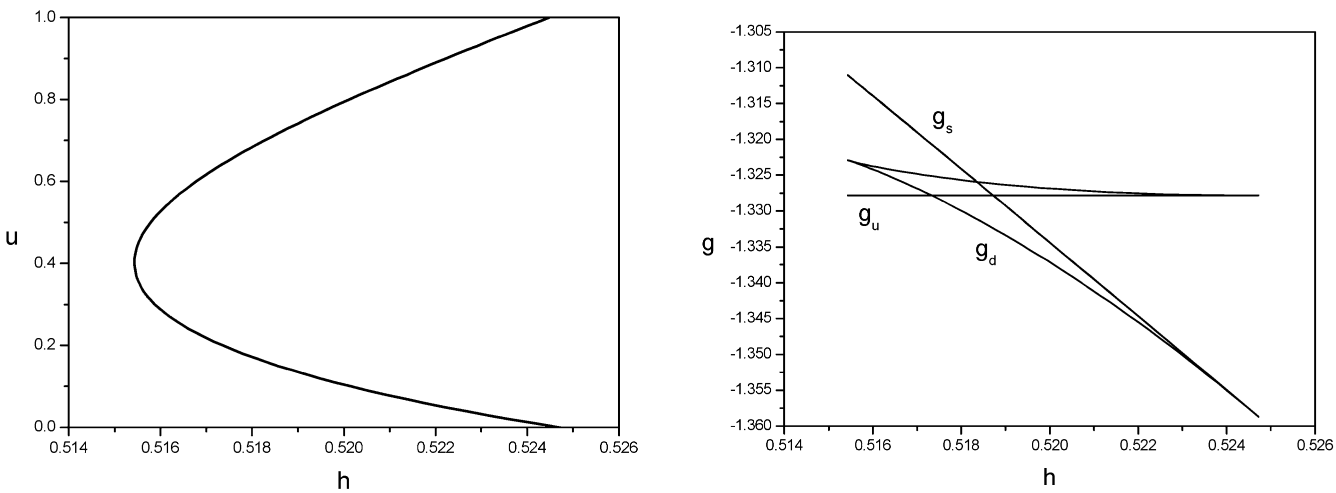
The results depicted in figure 9 are similar to those of figure 8, the uniform twisted state and the saturation state form the bistable state. When $\zeta'=-0.2$, $\lambda=20$, the results are similar to those of figure 5; the transition at threshold and saturation points is continuous and is of second order.

Numerical calculations applied to the Fréedericksz transition, produce the following results. (1) when $\zeta'<0$,



(*a*) (*b*)

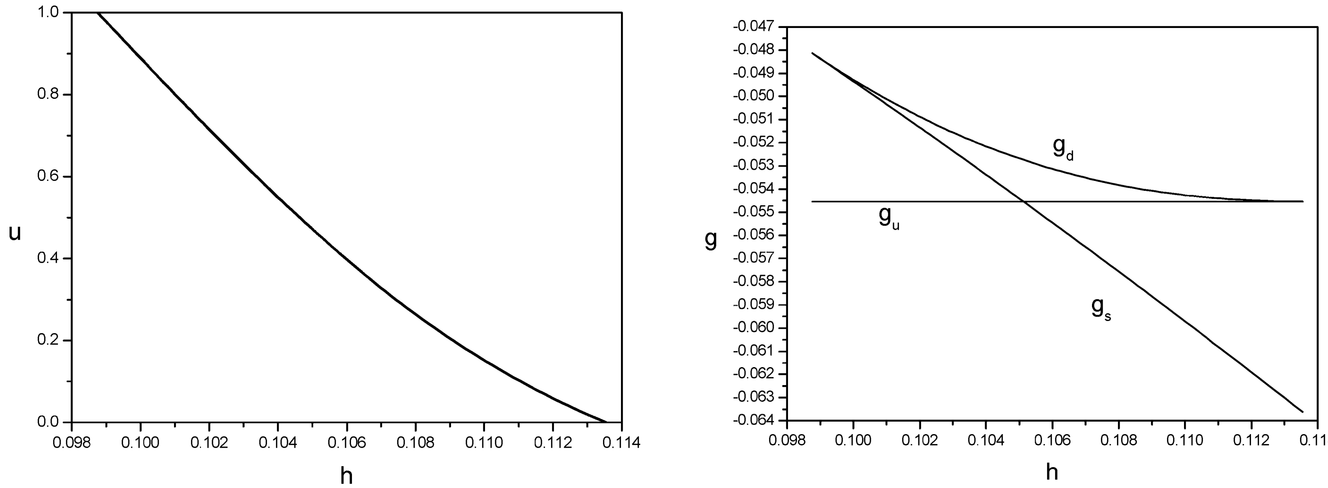
Figure 6. Plots of (*a*) *u* versus *h* and *g* versus *h*; $\zeta'=0.2$, $\lambda=0.77$.



(*a*) (*b*)

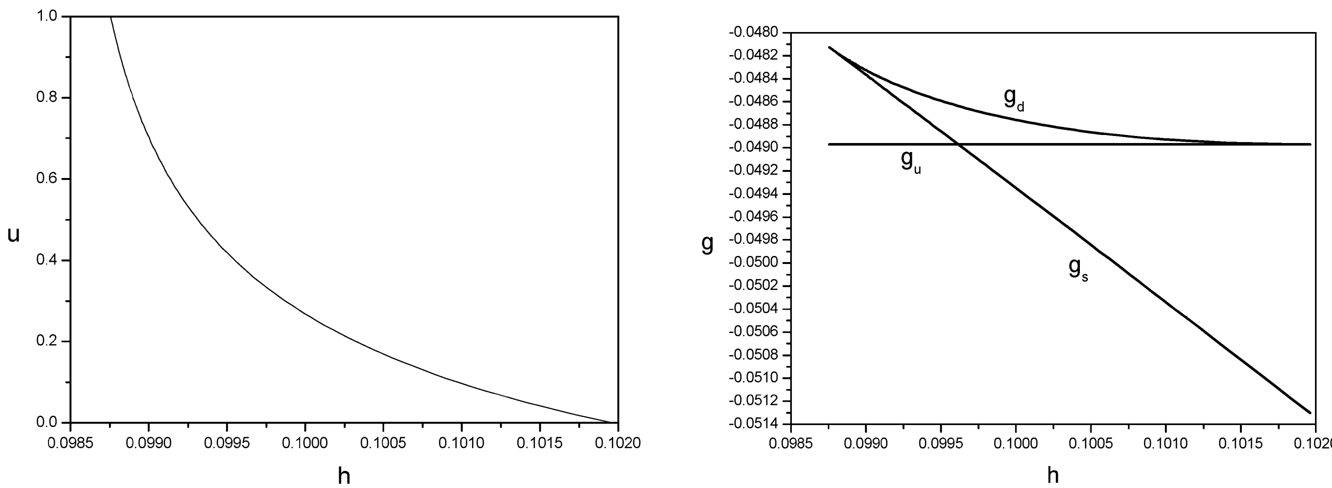
Figure 7. Plots of *u* versus *h* and *g* versus *h*; $\zeta'=0$, $\lambda=1.07$.

the threshold field strength is smaller than the saturation field strength, and the transition at the threshold and saturation points is second order. (2) When $\zeta'=0$, λ takes some value in the vicinity of the intersection of the threshold field strength and the saturation field strength, and the uniform twisted state and the disturbed state form the bistable state. The threshold field strength of the first order transition is calculated and is compared with the field strength of the second order transition in figure 2. As we can see, the threshold field strength for the first order transition is smaller, not only than the saturation field strength, but than the threshold field strength of the second order transition. This is consistent with previous results [11]. When $\zeta'=0.2$, λ takes some value in the vicinity of the intersection of the threshold field strength and the saturation field strength, the disturbed state and the saturation state form the bistable state. The saturation field strength of the first order transition is calculated and is compared with the field strength of the second order transition in figure 1. (3) When $\zeta'\geqslant 0$ the unusual case emerges; λ takes an arbitrary value, and the uniform twisted state and the saturation state can form a bistable state. The saturation field strength is



(*a*) (*b*)

Figure 8. Plots of (*a*) *u* versus *h* and (*b*) *g* versus *h*; $\zeta'=0.2$, $\lambda=20$.



(*a*) (*b*)

Figure 9. Plots of (*a*) *u* versus *h* and (*b*) *g* versus *h*; $\zeta'=0$, $\lambda=20$.

calculated, giving the dotted curves plotted in figures 1 and 2. In this situation, the threshold field strength is meaningless.

6. Conclusion

In this paper, we report a study of the bistable state of a weak anchoring NTLC cell. The results indicate that under different conditions, the uniform twisted state and the disturbed state, the disturbed state and the saturation state, and the saturation state and the uniform twisted state, can form bistable states; we calculate the threshold field strength and the saturation field strength. This may help in the design of a SBE cell with weak anchoring.

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